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LETTER TO THE EDITOR

Quantization of $U_q[osp(1/2n)]$ with deformed para-Bose operators

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Abstract. The observation that n pairs of para-Bose (pB) operators generate the universal enveloping algebra of the orthosymplectic Lie superalgebra $osp(1/2n)$ is used in order to define deformed pB operators. It is shown that these operators are an alternative to the Chevalley generators. On this background $U_q[osp(1/2n)]$, its 'Cartan-Weyl' generators and their 'supercommutation' relations are written down entirely in terms of deformed pB operators. An analogue of the Poincaré–Birkhoff–Witt theorem is formulated.

Soon after parastatistics was invented [1], it was realized that it carries a deep algebraic structure. More precisely, any n pairs of para-Fermi operators generate the simple Lie algebra $B_n \equiv so(2n+1)$ [2], whereas n pairs of para-Bose creation and annihilation operators (CAO's) $\hat{A}_1^{\pm}, \ldots, \hat{A}_n^{\pm}$ generate a Lie superalgebra [3], which is isomorphic to one of the basic Lie superalgebras (LSs) in the classification of Kac [4], namely to the orthosymplectic Lie superalgebra $osp(1/2n) \equiv B(0/n)$ [5]. In somewhat more implicit form the para-Bose operators (pB operators) were introduced even earlier by Wigner [6] in the search for the most general commutation relations between a position operator q and a momentum operator p of a one dimensional oscillator, so that the Heisenberg equations are identical with the Hamiltonian equations. The operators p, q turned out to generate $osp(1/2)$ and in fact Wigner was the first to find a class of (infinite-dimensional) representations of a Lie superalgebra [7]. Later on the results of Wigner gave rise to more general quantum systems (see [8] for references in this respect and for a general introduction to parastatistics), and in particular to quantum systems related to the classes $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ of basic LSs [9].

Purely algebraically the pB operators are defined as operators, which satisfy the relations ($\xi, \eta, \varepsilon = \pm$ or ± 1, $i, j, k = 1, 2, \ldots, n$; $[x, y] = xy - yx$, $\{x, y\} = xy + yx$)

$$[\{\hat{A}_i^{\xi}, \hat{A}_j^{\eta}\}, \hat{A}_k^{\varepsilon}] = (\varepsilon - \xi)\delta_{ik}\hat{A}_j^{\eta} + (\varepsilon - \eta)\delta_{jk}\hat{A}_i^{\xi}. \tag{1}$$

Let $L(n) = \text{lin env}\{\{\hat{A}_i^{\xi}, \hat{A}_j^{\eta}\}, \hat{A}_k^{\varepsilon} | \xi, \eta, \varepsilon = \pm, i, j, k = 1, 2, \ldots, n\}$ be a $\boldsymbol{Z}_2$ graded linear space with the pB operators as odd elements. Define a supercommutator on it as an anticommutator between any two odd elements and as a commutator otherwise.

Proposition 1 [5]. $L(n)$ is a LS, isomorphic to $osp(1/2n)$, with an odd subspace, spanned by the pB operators, and an even subalgebra $sp(2n) = \text{lin env}\{\{\hat{A}_i^{\xi}, \hat{A}_j^{\eta}\} | \xi, \eta, = \pm, i, j =$

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$1, 2, \ldots, n\}$. The pB operators define uniquely $osp(1/2n)$. The associative superalgebra with unity, the pB operators as free generators, and the relations (1) is the universal enveloping algebra $U[osp(1/2n)]$ of $osp(1/2n)$. The set of all (arbitrarily) ordered monomials of the operators

$$H_i = -\tfrac{1}{2}\{\hat{A}_i^-, \hat{A}_i^+\} \qquad \hat{A}_i^{\pm} \qquad \{\hat{A}_i^-, \hat{A}_j^+\} \qquad \{\hat{A}_i^{\xi}, \hat{A}_j^{\xi}\} \tag{2}$$
$$i \neq j = 1, \ldots, n \quad \xi = \pm$$

constitute a basis in $U[osp(1/2n)]$, whereas the operators (2) together with all $(\hat{A}_i^{\pm})^2$ define a basis, a Cartan–Weyl basis, in $osp(1/2n)$.

The proof is based on the relations, following from (1) and (2), namely

$$[H_i, \hat{A}_j^{\pm}] = \mp\delta_{ij}\hat{A}_j^{\pm} \qquad \{\hat{A}_i^-, \hat{A}_i^+\} = -2H_i \tag{3}$$

$$[\{\hat{A}_i^+, \hat{A}_j^-\}, \hat{A}_k^+] = \delta_{jk}\hat{A}_i^+ \qquad [\{\hat{A}_i^+, \hat{A}_j^-\}, \hat{A}_k^-] = \delta_{ik}\hat{A}_j^- \qquad i \neq j \tag{4}$$

$$[\{\hat{A}_i^{\xi}, \hat{A}_j^{\xi}\}, \hat{A}_k^{\xi}] = 0 \qquad i \neq j. \tag{5}$$

It is essential to point out that equations (3)–(5) define uniquely the supercommutation relations between all Cartan–Weyl generators. In other words, $U[osp(1/2n)]$ can be defined as an algebra with free generators H_i, $\hat{A}_i^{\pm}$, $i = 1, \ldots, n$ and the relations (3)–(5).

On the grounds of Proposition 1 and using the circumstance that $U[osp(1/2n)]$ has already been quantized, i.e., deformed to a superalgebra $U_q[osp(1/2n)] \equiv U_q$, which preserves its Hopf algebra structure [10, 11], one can introduce the concept of deformed pB operators. The deformation of U_q was carried out in a Chevalley basis. Clearly, it deforms all elements of U_q and in particular the pB operators $\hat{A}_1^{\pm}, \ldots, \hat{A}_n^{\pm}$. The purpose of the present letter is to give explicit relations for the deformed pB operators. We will show that the deformed pB operators define entirely the superalgebra. We will write down also the quantum analogue of the Cartan–Weyl generators in terms of pB operators and the supercommutation relations between the Cartan–Weyl generators (2). To our knowledge such relations for any n have not been given explicitly even in terms of the Chevalley generators. So far a deformation of pB operators was carried out for $n = 1$ [12] and $n = 2$ [13, 14] cases; there have been several other deformations, which were not endowed with a Hopf algebra structure [15, 16] (see also references therein).

We proceed to introduce U_q. The Cartan matrix (α_{ij}) is chosen as in [11], i.e., as $n \times n$ symmetric matrix with $\alpha_{nn} = 1$, $\alpha_{ii} = 2$, $i = 1, \ldots, n-1$, $\alpha_{j,j+1} = \alpha_{j+1,j} = -1$, $j = 1, \ldots, n-1$, and all other $\alpha_{ij} = 0$. Let $[x] = (q^x - q^{-x})/(q - q^{-1})$. Then U_q is the free associative superalgebra with Chevalley generators $e_i, f_i, k_i = q^{h_i}$, $i = 1, \ldots, n$, which satisfy the Cartan relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1 \qquad k_i k_j = k_j k_i \qquad i, j = 1, \ldots, n, \tag{6}$$

$$k_i e_j = q^{\alpha_{ij}} e_j k_i \qquad k_i f_j = q^{-\alpha_{ij}} f_j k_i \qquad i, j = 1, \ldots, n \tag{7}$$

$$\{e_n, f_n\} = \frac{k_n - k_n^{-1}}{q - q^{-1}} \qquad [e_i, f_j] = \delta_{ij}\frac{k_i - k_i^{-1}}{q - q^{-1}} = \delta_{ij}[h_i] \tag{8}$$
$$\forall i, j = 1, \ldots, \quad \text{except } i = j = n$$

the Serre relations for the simple positive root vectors

$$[e_i, e_j] = 0 \qquad \text{if } i, j = 1, \ldots, n \quad \text{and } |i - j| > 1 \tag{9}$$

$$e_i^2 e_{i+1} - (q+q^{-1}) e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0 \qquad i=1, \ldots, n-1 \tag{10}$$

$$e_i^2 e_{i-1} - (q+q^{-1}) e_i e_{i-1} e_i + e_{i-1} e_i^2 = 0 \qquad i=2, \ldots, n-1 \tag{11}$$

$$e_n^3 e_{n-1} + (1-q-q^{-1})(e_n^2 e_{n-1} e_n + e_n e_{n-1} e_n^2) + e_{n-1} e_n^3 = 0 \tag{12}$$

and the Serre relations obtained from (9)–(12) by replacing e_i with f_i everywhere. The grading on U_q is induced from the requirement that the generators e_n, f_n are odd and all other generators are even. Throughout $[a,b]_{q^n} = ab - (-1)^{\deg(a)\deg(b)} q^n ba$ and it is assumed that the deformation parameter q is any complex number except $q=0$, $q=1$ and $q^2=1$. Equations (6)–(12) are invariant with respect to the antiinvolution $(e_i)^* = f_i$, $(k_i)^* = k_i^{-1}$, $(q)^* = q^{-1}$.

The action of the coproduct Δ, the antipode S and the counit ε reads [11]

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i \qquad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i \qquad \Delta(k_i) = k_i \otimes k_i \tag{13}$$

$$S(e_i) = -k_i^{-1} e_i \qquad S(f_i) = -f_i k_i \qquad S(k_i) = k_i^{-1} \tag{14}$$

$$\varepsilon(e_i) = \varepsilon(f_i) = \varepsilon(k_i) = 0 \qquad \varepsilon(1) = 1. \tag{15}$$

Define $2n$ odd elements from $U_q[osp(1/2n)]$ as $(i=1, \ldots, n-1)$

$$A_i^- = -\sqrt{2}[e_i, [e_{i+1}, [e_{i+2}, [\ldots, [e_{n-2}, [e_{n-1}, e_n]_{q^{-1}}]_{q^{-1}} \ldots]_{q^{-1}} \tag{16}$$
$$A_n^- = -\sqrt{2} e_n$$

$$A_i^+ = \sqrt{2}[\ldots, [f_n, f_{n-1}]_q, f_{n-2}]_q, \ldots]_q, f_{i+2}]_q, f_{i+1}]_q, f_i]_q \qquad A_n^+ = \sqrt{2} f_n \tag{17}$$

and another n even 'Cartan' elements

$$K_i = k_i k_{i+1} \ldots k_n = q^{H_i} \qquad H_i = h_i + \ldots + h_n \qquad i=1, \ldots, n. \tag{18}$$

We call the operators $K_i^{\pm 1}, A_i^{\pm}$, $i=1, \ldots, n$ pre-oscillator generators, since in a certain representation of $U_q[osp(1/2n)]$ [17] these operators generate the oscillator algebra of the deformed Bose creation and annihilation operators as introduced in [18–20] or in [21]. The antiinvolution on them reads: $(A_i^{\pm})^* = -A_i^{\mp}$, $(K_i)^* = K_i^{-1}$, $i=1, \ldots, n$.

A substantial part of all computational time was spent in deriving the following relations:

$$K_i A_j^{\pm} = q^{\mp \delta_{ij}} A_j^{\pm} K_i \qquad \{A_i^-, A_i^+\} = -2 \frac{K_i - K_i^{-1}}{q - q^{-1}} \equiv -2[H_i] \qquad i=1, \ldots, n \tag{19}$$

$$[\{A_i^-, A_j^+\}, A_k^+] = 0 \qquad [\{A_i^-, A_j^+\}, A_j^+]_{q^{-1}} = 0$$
$$\text{if } i<j<k \quad \text{or } k<i<j$$

$$[\{A_i^-, A_j^+\}, A_i^+] = 2K_i A_j^+ \qquad [\{A_i^-, A_j^+\}, A_k^+] = (q^{-1} - q)\{A_i^-, A_k^+\} A_j^+$$
$$\text{if } i<k<j$$

$$[\{A_i^-, A_j^+\}, A_k^-] = 0 \qquad [\{A_i^-, A_j^+\}, A_i^-]_q = 0$$
$$\text{if } k<i<j \quad \text{or } i<j<k$$

$$[\{A_i^-, A_j^+\}, A_j^-] = -2K_j A_i^- \qquad [\{A_i^-, A_j^+\}, A_k^-] = (q - q^{-1})\{A_k^-, A_j^+\} A_i^-$$
$$\text{if } i<k<j \tag{20}$$

$$[\{A_i^-, A_j^+\}, A_k^-]=0 \qquad [\{A_i^-, A_j^+\}, A_i^-]_{q^{-1}}=0 \qquad \text{if } j<i<k \quad \text{or } k<j<i$$

$$[\{A_i^-, A_j^+\}, A_j^-]=-2K_j^{-1}A_i^- \qquad [\{A_i^-, A_j^+\}, A_k^-]=(q^{-1}-q)\{A_k^-, A_j^+\}A_i^- \qquad \text{if } j<k<i$$

$$[\{A_i^-, A_j^+\}, A_k^+]=0 \qquad [\{A_i^-, A_j^+\}, A_j^+]_q=0 \qquad \text{if } k<j<i \quad \text{or } j<i<k$$

$$[\{A_i^-, A_j^+\}, A_i^+]=2K_i^-A_j^- \qquad [\{A_i^-, A_j^+\}, A_k^+]=(q-q^{-1})\{A_i^-, A_k^+\}A_j^+ \qquad \text{if } j<k<i$$

$$[\{A_i^\xi, A_j^\xi\}, A_k^\xi]_{q^2}=0 \qquad [\{A_i^\xi, A_j^\xi\}, A_j^\xi]_q=0 \qquad \text{if } i<j<k \qquad \xi=\pm$$

$$[\{A_i^\xi, A_j^\xi\}, A_k^\xi]_{q^{-2}}=0 \qquad [\{A_i^\xi, A_j^\xi\}, A_j^\xi]_{q^{-1}}=0 \qquad \text{if } k<i<j \qquad \xi=\pm \tag{21}$$

$$[\{A_i^\xi, A_j^\xi\}, A_k^\xi]=0 \qquad \text{if } i<k<j \qquad \xi=\pm.$$

To this end the following lemma turned out to be particularly useful.

Lemma 1. If $[a, b]=0$, then $[a, [c, b]_q]_q=[[a, c]_q, b]_q$, $[a, [c, b]_{q^{-1}}]_q=[[a, c]_q, b]_{q^{-1}}$; if a, b are even elements and $[a, c]=0$, then $(q+q^{-1})[b, [a, [b, c]_q]_q]=[a, [b, [b, c]_q]_{q^{-1}}]_{q^2}-[[b, [b, a]_q]_{q^{-1}}, c]_{q^2}$.

In particular, if $[a_i, a_j]=0$ for all $|i-j|>1$, then

$$[a_k, [a_{k-1}, [a_{k-2}[\ldots, [a_3, [a_2, a_1]_q\ldots]_q =[\ldots[a_k, a_{k-1}]_q, a_{k-2}]_q, \ldots]_q, a_3]_q, a_2]_q, a_1]_q. \tag{22}$$

Proposition 2. (A) The free algebra $\hat{U}_q$ of the pre-oscillator generators and the relations (19)–(21) is the quantized $U_q[osp(1/2n)]$ algebra with generators $e_i, f_i, k_i=q^{h_i}$, $i=1, \ldots, n$, and relations (6)–(12); (B) The operators

$$K_i^{\pm 1} \qquad A_i^\pm \qquad \{A_i^-, A_j^+\} \qquad \{A_i^\xi, A_j^\xi\} \qquad i\neq j=1, \ldots, n \tag{23}$$

are the analogue of the Cartan–Weyl generators (2); (C) The relations (19)–(21) are the analogue of the supercommutation relations (3), (4), (5) among all Cartan–Weyl generators (2); (D) The set of all normally ordered monomials [11] of the Cartan–Weyl generators (23) constitute a basis in $U_q[osp(1/2n)]$ (Poincaré–Birkhoff–Witt theorem).

We sketch the proof. By construction $\hat{U}_q$ is a subalgebra of $U_q[osp(1/2n)]$. The expressions of the Chevalley generators in terms of the pre-oscillator generators read $(i=1, \ldots, n-1)$:

$$e_n=-(2)^{-1/2}A_n^- \qquad f_n=(2)^{-1/2}A_n^+ \qquad e_i=\frac{q}{2}\{A_i^-, A_{i+1}^+\}K_{i+1}^{-1}$$
$$f_i=\frac{1}{2q}\{A_i^+, A_{i+1}^-\}K_{i+1}. \tag{24}$$

From (24) and (19)–(21) one derives all relations (6)–(12) among the Chevalley generator. Hence $U_q[osp(1/2n)]$ is a subalgebra of $\hat{U}_q$ and therefore $U_q[osp(1/2n)] = \hat{U}_q$. This proves (A). From (16)–(18) one concludes that in the limit $q \to 1$ the relations (19), (20), (21) of the operators (23) reduce to the relations (3), (4), (5) of the Cartan–Weyl generators (2), which proves (B) and (C). The normal order between the groups of the positive root vectors (PRV) A_i, $\{A_i^-, A_j^\pm\}$, $i<j$, the negative root vectors (NRV) and the Cartan generators K_i is arbitrary but fixed. For instance, PRV < NRV < K_i. The order among the PRV can be taken to be

$$\{A_i^-, A_p^+\} < A_i^- < \{A_i^-, A_q^-\} < \{A_j^-, A_r^+\} < A_j^- < \{A_j^-, A_s^-\}$$

for all $i<p, q$; $i<j$; $j<r, s$ and similarly for the NRV. The possibility for such ordering and hence the proof of (D) follows directly from the relations (19)–(21).

Remark. The relations (19)–(21) are by no means the minimal set of relation among the pre-oscillator generators, which define $U_q[osp(1/2n)]$. On the contrary, they are the analogue of all supercommutation relations among the generators in the non-deformed case.†

We observe that the Cartan–Weyl generators are expressed in an easy way in terms of the pre-oscillator generators and even the anticommutators in (23) remain undeformed. This is due to the fact that the roots of $A_1^+, \ldots, A_n^+$ are orthogonal to each other. Certainly, from (16)–(18) and (23) one can write down all Cartan–Weyl generators also in terms of the Chevalley generators, but the expressions are not so simple. Here are the Cartan–Weyl generators of the Hopf subalgebra $U_q[gl(n)]$:

$$\{A_i^-, A_j^+\} = -2[\ldots, [f_{i-1}, f_{i-2}]_q, f_{i-3}]_q, \ldots]_q, f_{j+2}]_q, f_{j+1}]_q, f_j]_q k_i^{-1} k_{i+1}^{-1} \ldots k_n^{-1} \qquad i>j \tag{25}$$

and the conjugate of (25) with respect to the antiinvolution introduced previously.

In conclusion we should come back to para-Bose statistics. We have seen that at $q \to 1$ the relations (19), (20), (21) reduce to (3), (4), (5) and, hence to the equations (1), defining the pB operators. Therefore the operators $A_1^\pm, \ldots, A_n^\pm$ are deformed para-Bose operators.

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† After the work on the present investigation had been completed, we received a preprint from Hadjiivanov [22], where the question about a minimal set of relations of the pB operators, defining $U_q[osp(1/2n)]$, is settled.

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